2022.M29

2022L003A1EL

Coimisiún na Scrúduithe Stáit
State Examinations Commission

Leaving Certificate Examination 2022

Mathematics

Paper 1

Higher Level

Friday 10 June   Afternoon 2:00 – 4:30

220 marks

Examination Number

Day and Month of Birth

For example, 3rd February is entered as 0302

Centre Stamp

Do not write on this page

Instructions

There are **two** sections in this examination paper.

Section A	Concepts and Skills	120 marks	6 questions
Section B	Contexts and Applications	100 marks	4 questions

Answer questions as follows:
- any **four** questions from Section A – Concepts and Skills
- any **two** questions from Section B – Contexts and Applications.

Write your Examination Number in the box on the front cover.

Write your answers in blue or black pen. You may use pencil in graphs and diagrams only.

This examination booklet will be scanned and your work will be presented to an examiner on screen. Anything that you write outside of the answer areas may not be seen by the examiner.

Write all answers into this booklet. There is space for extra work at the back of the booklet. If you need to use it, label any extra work clearly with the question number and part.

The superintendent will give you a copy of the *Formulae and Tables* booklet. You must return it at the end of the examination. You are not allowed to bring your own copy into the examination.

You will lose marks if your solutions do not include relevant supporting work.

You may lose marks if the appropriate units of measurement are not included, where relevant.

You may lose marks if your answers are not given in simplest form, where relevant.

Write the make and model of your calculator(s) here:

Section A Concepts and Skills 120 marks

Answer **any four questions** from this section.

Question 1 **(30 marks)**

(a) Find the two values of $m \in \mathbb{Z}$ for which the following equation in x has exactly **one** solution:

$$3x^2 - mx + 3 = 0$$

(b) Explain why the following equation in x has **no** real solutions:

$$(2x + 3)^2 + 7 = 0$$

(c) **(i)** Show that $x = -1$ is **not** a solution of $3x^2 + 2x + 5 = 0$.

(ii) Find the **remainder** when $3x^2 + 2x + 5$ is divided by $x + 1$.

That is, find the value of c when $3x^2 + 2x + 5$ is written in the form

$$3x^2 + 2x + 5 = (x + 1)(ax + b) + c$$

where $a, b, c \in \mathbb{Z}$.

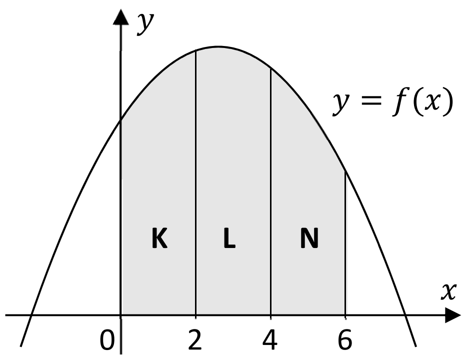
Remainder, $c =$ ____________

Question 2 **(30 marks)**

(a) $g(x) = 2x^2 + 5x + 6$, where $x \in \mathbb{R}$.

Find $\int g(x)\ dx$.

(b) The diagram shows the graph of a function $f(x) = ax^2 + bx + c$, where $a, b, c \in \mathbb{Z}$.
Three regions on the diagram are marked **K**, **L**, and **N**.
Each of these regions is bounded by the x-axis, the graph of $f(x)$, and two vertical lines.



(i) The area of region **K** is 538 square units. Use integration of $f(x)$ to show that:

$$4a + 3b + 3c = 807$$

(ii) The areas of the three regions **K**, **L**, and **N** give the following three equations (including the equation from **part (b)(i)**):

$$4a + 3b + 3c = 807$$
$$28a + 9b + 3c = 879$$
$$76a + 15b + 3c = 663$$

Solve these equations to find the values of a, b, and c.

$a =$ ____________ $b =$ ____________ $c =$ ____________

Question 3 **(30 marks)**

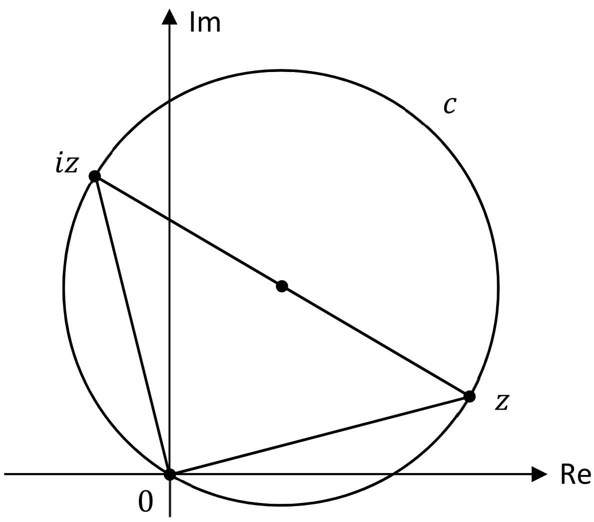
(a) $z = 6 + 2i$, where $i^2 = -1$.

(i) Show that $z - iz = 8 - 4i$.

(ii) Show that $|z|^2 + |iz|^2 = |z - iz|^2$

(iii) The circle c passes through the points z, iz, and 0, as shown in the diagram below (not to scale). z and iz are endpoints of a diameter of the circle.

Find the area of the circle c in terms of π.
There is space on the next page for your solution.



Space for ***part (a)(iii)***.

(b) $\left(\sqrt{3} - i\right)^9$ can be written in the form $a + ib$, where $a, b \in \mathbb{Z}$ and $i^2 = -1$.

Use de Moivre's Theorem to find the value of a and the value of b.

$a =$ ____________ $b =$ ____________

Question 4 **(30 marks)**

(a) A sequence u_1 , u_2 , u_3 , ... is defined as follows, for $n \in \mathbb{N}$:

$$u_1 = 2, \quad u_2 = 64, \quad u_{n+1} = \sqrt{\frac{u_n}{u_{n-1}}}$$

Write u_3 in the form 2^p , where $p \in \mathbb{R}$.

(b) The first three terms in an **arithmetic** sequence are as follows, where $k \in \mathbb{R}$:

$$5e^{-k}, \quad 13, \quad 5e^{k}$$

(i) By letting $y = e^{k}$ in this arithmetic sequence, show that:

$$5y^2 - 26y + 5 = 0$$

(ii) Use the equation in y in **part (b)(i)** to find the two possible values of k.
Give each value in the form $\ln p$ or $-\ln p$, where $p \in \mathbb{N}$.

Question 5 **(30 marks)**

(a) $g(x) = x^2 - \frac{1}{x}$ where $x \in \mathbb{R}$.

Find $g'(x)$, the derivative of $g(x)$.

(b) $f(x) = 2x^3 - 21x^2 + 40x + 63,$ where $x \in \mathbb{R}$.

(i) $x + 1$ is a factor of $f(x)$. Find the three values of x for which $f(x) = 0$.

$x =$ ____________ , ____________ , or ____________

(ii) Find the range of values of x for which $f'(x)$ is negative, correct to 2 decimal places.

Question 6 **(30 marks)**

(a) Differentiate $f(x) = 2x^2 + 4x$ with respect to x, from first principles.

(b) A rectangle is expanding in area. Its width is x cm, where $x \in \mathbb{R}$ and $x > 0$.
Its length is always four times its width.

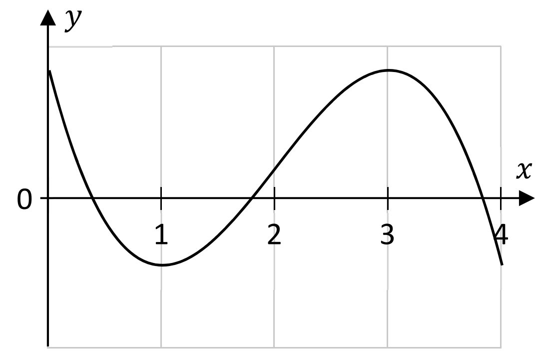
Find the rate of change of the area of the rectangle with respect to its width, x, when the area of the rectangle is 225 cm^2.

There is space for more work on the next page.

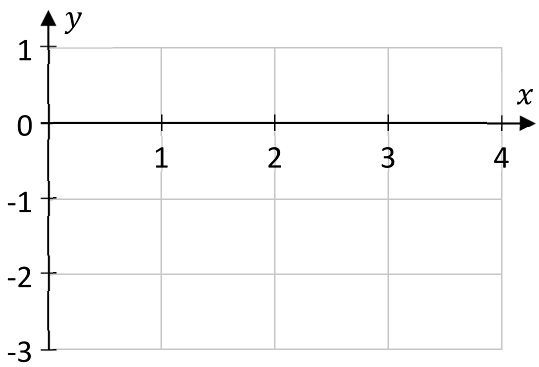
(c) The graph of a cubic function $p(x)$ is shown in the first diagram below, for $0 \leq x \leq 4$, $x \in \mathbb{R}$. The maximum value of $p'(x)$ in this domain is 1, and $p'(0) = -3$, where $p'(x)$ is the derivative of $p(x)$.

Use this information to draw the graph of $p'(x)$ on the second set of axes below, for $0 \leq x \leq 4$, $x \in \mathbb{R}$.

Graph of $y = p(x)$



Graph of $y = p'(x)$



Section B	Contexts and Applications	100 marks

Answer **any two questions** from this section.

Question 7 **(50 marks)**

Hannah is doing a training session. During this session, her heart-rate, $h(x)$, is measured in beats per minute (BPM), where x is the time in minutes from the start of the session, $x \in \mathbb{R}$.
For the first 8 minutes of the session, Hannah does a number of exercises.
As she does these exercises, her heart-rate changes. In this time, $h(x)$ is given by:

$$h(x) = 2x^3 - 28{\cdot}5x^2 + 105x + 70$$

(a) Work out Hannah's heart-rate 4 minutes after the start of the session.

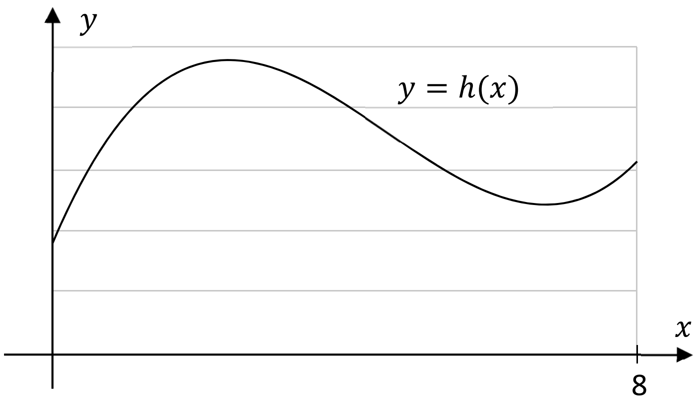
(b) Find $h'(x)$.

(c) Find $h'(2)$, **and** explain what this value means in the context of Hannah's heart-rate.

$h'(2)$:

Explanation:

The graph below shows $y = h(x)$, where $0 \leq x \leq 8$, $x \in \mathbb{R}$.



(d) Find the least value and the greatest value of $h(x)$, for $0 \leq x \leq 8$, $x \in \mathbb{R}$.
Use calculus in your solution. You may also use information from the graph above, which is to scale.

Least value of $h(x)$: ____________ Greatest value of $h(x)$: ____________

This question continues on the next page.

(e) How long after the start of the session is Hannah's heart-rate decreasing most quickly, within the first 8 minutes? Give your answer in minutes and seconds.

Remember that $h(x) = 2x^3 - 28{\cdot}5x^2 + 105x + 70$.

Bruno, Karen, and Martha start a training session at the same time as Hannah.
All of their heart-rates are measured in BPM.

(f) **(i)** For the first 8 minutes of the session, Bruno's heart-rate, $b(x)$, is always 15 BPM more than Hannah's heart-rate.

Use this information to write $b'(x)$ **in terms of $\boldsymbol{h'(x)}$**, where $0 \le x \le 8$, $x \in \mathbb{R}$.

(ii) For the first 8 minutes of the session, Karen's heart-rate, $k(x)$, is always 10% less than Hannah's heart-rate.

Use this information to write $k'(x)$ **in terms of $\boldsymbol{h'(x)}$**, where $0 \le x \le 8$, $x \in \mathbb{R}$.

(g) Martha does each exercise for a longer time than Hannah.
For $0 \leq x \leq 10$, Martha's heart-rate, $m(x)$, is:

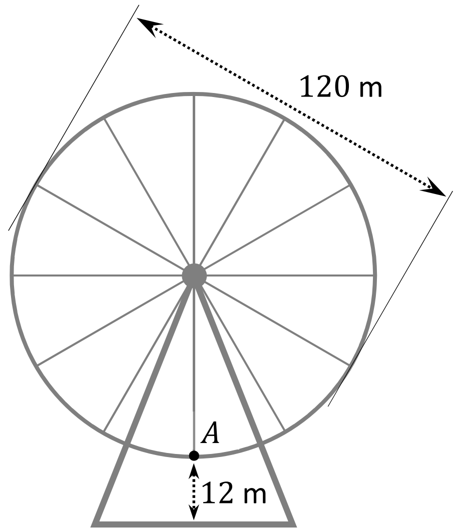
$$m(x) = h(0{\cdot}8\,x)$$

Use $h(x) = 2x^3 - 28{\cdot}5x^2 + 105x + 70$ to write $m(x)$ in the form:

$$m(x) = ax^3 + bx^2 + cx + d$$

where $a, b, c, d \in \mathbb{R}$, for $0 \leq x \leq 10$.

Question 8 **(50 marks)**



A Ferris wheel has a diameter of 120 m.
When it is turning, it completes exactly 10 full rotations in one hour.
The diagram above shows the Ferris wheel before it starts to turn.
At this stage, the point A is the lowest point on the circumference of the wheel, and it is at a height of 12 m above ground level.

The height, h, of the point A after the wheel has been turning for t minutes is given by:

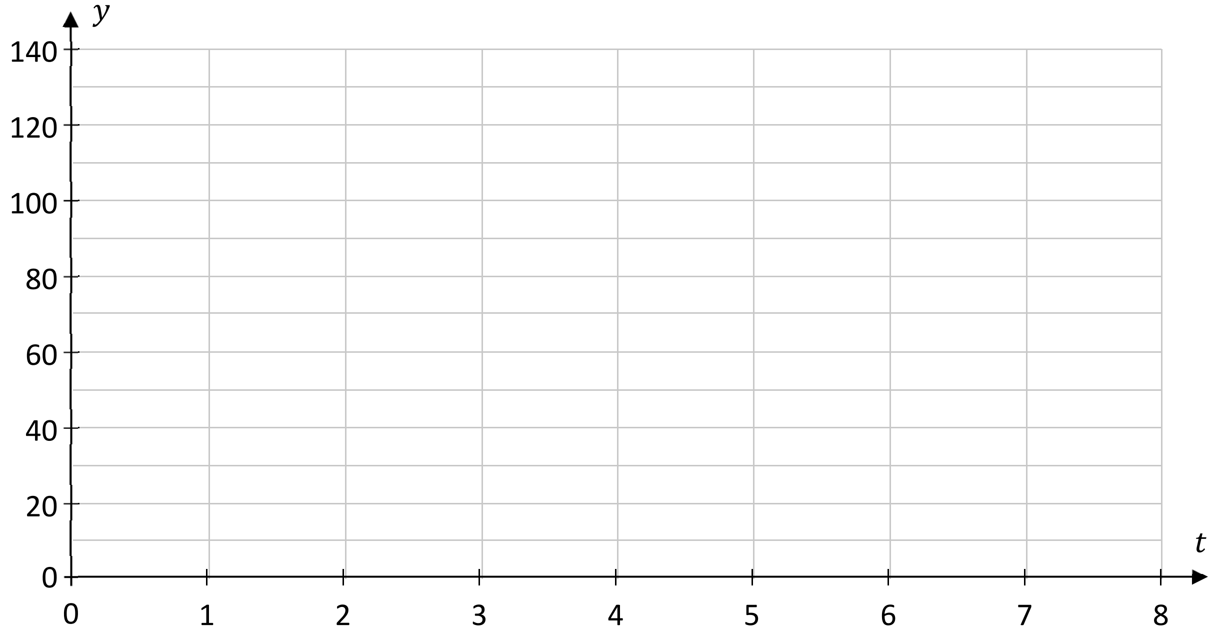
$$h(t) = 72 - 60\cos\left(\frac{\pi}{3}t\right)$$

where h is in metres, $t \in \mathbb{R}$, and $\frac{\pi}{3}t$ is in radians.

(a) Complete the table below. The value of $h(1)$ is given.

t	0	1	2	3	4	5	6	7	8
$h(t)$		42							

(b) Draw the graph of $y = h(t)$ for $0 \leq t \leq 8$, $t \in \mathbb{R}$.



(c) Find the period and range of $h(t)$.

Period = ________________ Range = [,]

(d) During a 50-minute period, what is the **greatest** number of minutes for which the point A could be higher than 42 m?

This question continues on the next page.

(e) By solving the following equation, find the **second** time (value of t) that the point A is at a height of 110 m, after it starts turning:

$$72 - 60\cos\left(\frac{\pi}{3}t\right) = 110$$

Give your answer in minutes, correct to 2 decimal places.

(f) Use integration to find the average height of the point A over the first 8 minutes that the wheel is turning. Give your answer correct to 1 decimal place.

Remember that $h(t) = 72 - 60\cos\left(\frac{\pi}{3}t\right)$.

Question 9 **(50 marks)**

Alex gets injections of a medicinal drug. Each injection has 15 mg of the drug.
Each day, the amount of the drug left in Alex's body from an injection **decreases** by 40%.
So, the amount of the drug (in mg) left in Alex's body t days after a single injection is given by:

$$15(0{\cdot}6)^t$$

where $t \in \mathbb{R}$.

(a) Find the amount of the drug left in Alex's body $2{\cdot}5$ days after a single 15 mg injection. Give your answer in mg, correct to 2 decimal places.

(b) How long after a single 15 mg injection will there be exactly 1 mg of the drug left in Alex's body? Give your answer in days, correct to 1 decimal place.

Alex is given a 15 mg injection of the drug at the same time **every** day for a long period of time.

(c) Explain why the total amount of the drug, in mg, in Alex's body immediately after the 4th injection is given by:

$$15 + 15(0{\cdot}6) + 15(0{\cdot}6)^2 + 15(0{\cdot}6)^3$$

(d) Find the total amount of the drug in Alex's body immediately after the 10th injection. Give your answer in mg, correct to 2 decimal places.

(e) Use the formula for the sum to infinity of a geometric series to estimate the amount of the drug (in mg) in Alex's body, after a long period of time during which he gets daily injections.

This question continues on the next page.

(f) Jessica also gets daily injections of a medicinal drug at the same time every day.
She gets d mg of the drug in each injection, where $d \in \mathbb{R}$.
Each day, the amount of the drug left in Jessica's body from an injection **decreases** by 15%.

(i) Use the sum of a geometric series to show that the total amount of the drug (in mg) in Jessica's body immediately after the nth injection, where $n \in \mathbb{N}$, is:

$$\frac{20d(1 - 0{\cdot}85^n)}{3}$$

(ii) Immediately after the 7th injection, there are 50 mg of the drug in Jessica's body.

Find the amount of the drug in one of Jessica's daily injections.
Give your answer correct to the nearest mg.

Question 10 **(50 marks)**

A student is asked to memorise a long list of digits, and then write down the list some time later. The proportion, P, of the digits recalled correctly after t hours can be modelled by the function:

$$P(t) = 0{\cdot}82 - 0{\cdot}12\ln(t+1)$$

for $0 \leq t \leq 12$ and $t \in \mathbb{R}$.

(a) Find the proportion of the digits recalled correctly after 3 hours, according to this model. Give your answer correct to 2 decimal places.

(b) After how many hours would exactly 55% of the digits be recalled correctly, according to this model? Give your answer correct to 2 decimal places.

(c) **(i)** Find the value of $P'(1)$.

(ii) $P'(t)$ is always negative for $0 \leq t \leq 12$, $t \in \mathbb{R}$. What does this tell you about the proportion of digits recalled correctly after t hours, according to this model?

(d) Use calculus to show that the graph of $y = P(t)$ has **no** points of inflection, for $0 \leq t \leq 12$, $t \in \mathbb{R}$.

This question continues on the next page.

If we learn a skill, and then don't practise it, how well we can do it usually decreases over time. For example, if you learn to play the guitar, and then don't play the guitar for a number of months, you will probably not be as good at playing the guitar the first time you try it again.

This effect can be modelled by the following equation:

$$A = B(t+1)^c$$

where A is a measure of how well the skill can be done at a certain time ($t = 0$);
B is a measure of how well the skill can be done t months later, without practising;
c is a constant; and $A, B, c, t \in \mathbb{R}$.

(e) **(i)** Write c in terms of $\log_{10} A$, $\log_{10} B$, and $\log_{10}(t+1)$.

(ii) A student got 80% on a guitar exam.
After two **years** of not playing the guitar, the student got 47% on the same exam.

Use this to find the value of c in the model above, correct to 3 decimal places.

Page for extra work.
Label any extra work clearly with the question number and part.

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