Coimisiún na Scrúduithe Stáit
State Examinations Commission

LEAVING CERTIFICATE EXAMINATION, 2010

MATHEMATICS – HIGHER LEVEL

PAPER 2 (300 marks)

Monday, 14 June – MORNING, 9:30 to 12:00

Attempt **FIVE** questions from **Section A** and **ONE** question from **Section B**.
Each question carries 50 marks.

WARNING: **Marks will be lost if all necessary work is not clearly shown.**

Answers should include the appropriate units of measurement, where relevant.

SECTION A

Answer FIVE questions from this section.

1. **(a)** A circle with centre $(3, -4)$ passes through the point $(7, -3)$.
Find the equation of the circle.

(b) **(i)** Find the centre and radius of the circle $x^2 + y^2 - 8x - 10y + 32 = 0$.

(ii) The line $3x + 4y + k = 0$ is a tangent to the circle $x^2 + y^2 - 8x - 10y + 32 = 0$.
Find the two possible values of k.

(c) A circle has the line $y = 2x$ as a tangent at the point $(2, 4)$. The circle also passes through the point $(4, -2)$. Find the equation of the circle.

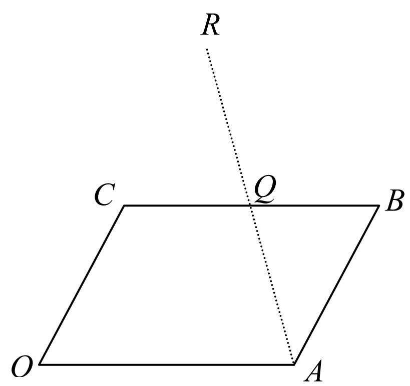
2. **(a)** A, B and C are points and O is the origin.
$\vec{a} = 2\vec{i} + 3\vec{j}$, $\vec{b} = -3\vec{i} - 6\vec{j}$, and $\overrightarrow{AC} = \overrightarrow{OB}$.
Express $\vec{c}$ in terms of $\vec{i}$ and $\vec{j}$.

(b) $\vec{u} = 2\vec{i} + \vec{j}$ and $\vec{v} = -\vec{i} + k\vec{j}$ where $k \in \mathbb{R}$.

(i) Express $|\vec{v}|$ and $\vec{u} \cdot \vec{v}$ in terms of k.

(ii) Given that $\cos\theta = -\dfrac{1}{\sqrt{2}}$, where θ is the angle between $\vec{u}$ and $\vec{v}$,
find the two possible values of k.

(c) $OABC$ is a parallelogram, where O is the origin.
Q is the midpoint of $[BC]$.
$[AQ]$ is extended to R such that $|AQ| = |QR|$.



(i) Express $\vec{q}$ in terms of $\vec{a}$ and $\vec{c}$.

(ii) Express $\overrightarrow{AQ}$ in terms of $\vec{a}$ and $\vec{c}$.

(iii) Show that the points O, C and R are collinear.

3. **(a)** The line $3x + 4y - 7 = 0$ is perpendicular to the line $ax - 6y - 1 = 0$.
Find the value of a.

(b) **(i)** The line $4x - 5y + k = 0$ cuts the x-axis at P and the y-axis at Q.
Write down the co-ordinates of P and Q in terms of k.

(ii) The area of the triangle OPQ is 10 square units, where O is the origin.
Find the two possible values of k.

(c) f is the transformation $(x, y) \to (x', y')$, where $x' = x + y$ and $y' = x - y$.
The line l has equation $y = mx + c$.

(i) Find the equation of $f(l)$, the image of l under f.

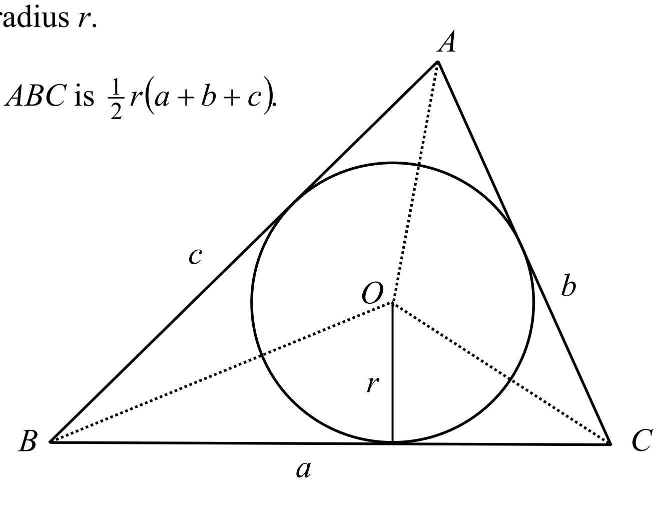
(ii) Find the value(s) of m for which $f(l)$ makes an angle of 45° with l.

4. **(a)** The area of a triangle PQR is 20 cm^2. $|PQ| = 10$ cm and $|PR| = 8$ cm.
Find the two possible values of $|\angle QPR|$.

(b) Find all the solutions of the equation $\cos 2x = \cos x$ in the domain $0^\circ \leq x \leq 360^\circ$.

(c) ABC is a triangle with sides of lengths a, b and c, as shown.
Its incircle has centre O and radius r.

(i) Show that the area of ΔABC is $\frac{1}{2}r(a + b + c)$.

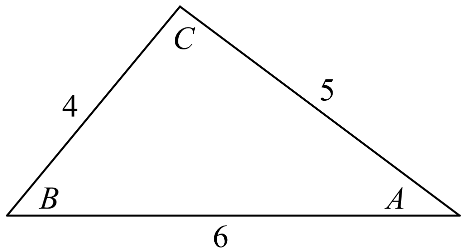


(ii) The lengths of the sides of a triangle are $a = p^2 + q^2$, $b = p^2 - q^2$, and $c = 2pq$, where p and q are natural numbers and $p > q$.
Show that this triangle is right-angled.

(iii) Show that the radius of the incircle of the triangle in part **(ii)** is a whole number.

5. **(a)** Given that $\tan\theta = \frac{1}{3}$, show that $\tan 2\theta = \frac{3}{4}$.

(b) A triangle has sides of lengths 4, 5 and 6.
The angles of the triangle are A, B and C, as in the diagram.



(i) Using the cosine rule, show that $\cos A + \cos C = \frac{7}{8}$.

(ii) Show that $\cos(A + C) = -\frac{9}{16}$.

(c) **(i)** Show that $(\cos A + \cos B)^2 + (\sin A + \sin B)^2 = 2 + 2\cos(A - B)$.

(ii) Hence solve the equation

$$(\cos 4x + \cos x)^2 + (\sin 4x + \sin x)^2 = 2 + 2\sqrt{3}\sin 3x$$

in the domain $0^\circ \le x \le 360^\circ$.

6. **(a)** One bag contains four red discs and six blue discs.
Another bag contains five red discs and seven yellow discs.
One disc is drawn from each bag.
What is the probability that both discs are red?

(b) α and β are the roots of the quadratic equation $px^2 + qx + r = 0$.
$u_n = l\alpha^n + m\beta^n$, for all $n \in \mathbb{N}$.
Prove that $pu_{n+2} + qu_{n+1} + ru_n = 0$, for all $n \in \mathbb{N}$.

(c) In a café there are 11 seats in a row at the counter.
Six people are seated at random at the counter.

How much more likely is it that all six are seated together than that no two of them are seated together?

7. **(a)** A password for a website consists of capital letters A, B, C,. … Z and/or digits 0, 1, 2, …. 9.

The password has four such characters and starts with a letter. For example, BA7A, C999 and DGKK are allowed, but 7DCA is not.

Show that there are more than a million possible passwords.

(b) Karen is about to sit an examination at the end of an English course. The course has twenty prescribed texts. Six of these are novels, four are plays and ten are poems.

The examination consists of a question on one of the novels, a question on one of the plays and a question on one of the poems.

Karen has studied four of the novels, three of the plays and seven of the poems.

Find the probability that:

(i) Karen has studied all three of the texts on the examination

(ii) Karen has studied none of the texts on the examination

(iii) Karen has studied at least two of the texts on the examination.

(c) The real numbers a, $2a$, $3a$, $4a$ and $5a$ have mean μ and standard deviation σ.

(i) Express μ and σ in terms of a.

(ii) Hence write down, in terms of a, the mean and the standard deviation of

$$3a+5,\ 6a+5,\ 9a+5,\ 12a+5,\ 15a+5.$$

SECTION B

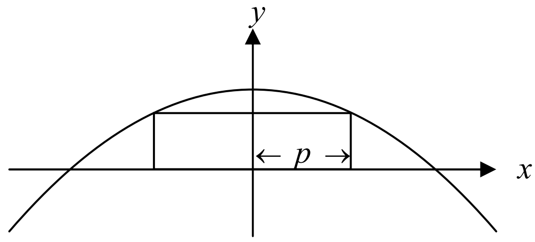
Answer one question from this section

8. **(a)** Use integration by parts to find $\int \log_e x \, dx$.

(b) A rectangle is inscribed between the curve $y = 9 - x^2$ and the x-axis, as shown.

(i) Write an expression for the area of the rectangle in terms of p.

(ii) Hence, calculate the area of the largest possible rectangle.



(c) **(i)** Derive the Maclaurin series for $f(x) = \cos x$, up to and including the term containing x^6.

(ii) Hence, and using the identity $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$, show that the first three non-zero terms of the Maclaurin series for $\sin^2 x$ are $x^2 - \frac{x^4}{3} + \frac{2x^6}{45}$.

(iii) Use these terms to find an approximation for $\sin^2\left(\frac{1}{2}\right)$, as a fraction.

9. **(a)** Z is a random variable with standard normal distribution.
Find $P(-1 < Z \le 1)$.

(b) A test consists of twenty multiple-choice questions. Each question has four possible answers, only one of which is correct. Seán decides to guess all the answers at random.

Find the probability that:

(i) Seán gets none of the answers correct

(ii) Seán gets exactly five of the answers correct

(iii) Seán gets four, five or six of the answers correct.

Give each of your answers correct to three decimals places.

(c) A bakery produces muffins. A random sample of 50 muffins is selected and weighed. The mean weight of the sample is 80 grams and the standard deviation is 6 grams.

Form a 95% confidence interval for the mean weight of muffins produced by the bakery.

10. **(a)** The binary operation $*$ is defined by $x * y = x + y - xy$, where $x, y \in \mathbb{R} \setminus \{-1\}$.

(i) Find the identity element.

(ii) Express x^{-1}, the inverse of x, in terms of x.

(b) G is the set of permutations of $\{1, 2, 3\}$ and the six elements of G are as follows:

$$a = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} \qquad b = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} \qquad c = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$$

$$d = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} \qquad f = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} \qquad g = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}.$$

$(G, \circ)$ is a group, where $\circ$ denotes composition.

(i) Write down b^{-1} and d^{-1}, the inverses of b and d respectively.

(ii) Verify that $(b \circ d)^{-1} = d^{-1} \circ b^{-1}$.

(iii) Write down the subgroups of $(G, \circ)$ of order 2.

(iv) K is the subgroup of $(G, \circ)$ of order 3. List the elements of K.

(v) H is a group under multiplication, where $H = \{1, w, w^2\}$ and $w^3 = 1$.
Give an isomorphism ϕ from $(K, \circ)$ to $(H, \times)$, justifying fully that it is an isomorphism.

11. **(a)** An ellipse with centre $(0, 0)$ has eccentricity $\frac{4}{5}$ and the length of its major axis is 2 units. Find its equation.

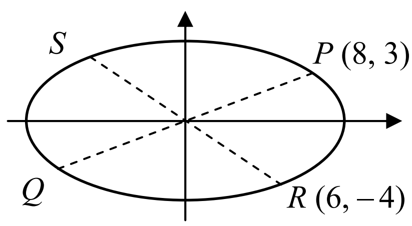
(b) f is an affine transformation. The point M is the mid-point of the line segment $[AB]$.

(i) Show that $f(M)$ is the mid-point of the line segment $[f(A)f(B)]$.

(ii) A triangle ABC has centroid G.
Show that the triangle $f(A)f(B)f(C)$ has centroid $f(G)$.

(c) An ellipse e has equation $\dfrac{x^2}{100} + \dfrac{y^2}{25} = 1$.

$[PQ]$ and $[RS]$ are diameters of the ellipse, where P is $(8, 3)$ and R is $(6, -4)$.



(i) Using a transformation to or from the unit circle, or otherwise, show that the diameters $[PQ]$ and $[RS]$ are conjugate.

(ii) Find the area of the parallelogram that circumscribes the ellipse at the points P, S, Q, and R.

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