2008. M30

Coimisiún na Scrúduithe Stáit
State Examinations Commission

LEAVING CERTIFICATE EXAMINATION, 2008

MATHEMATICS – HIGHER LEVEL

PAPER 2 (300 marks)

MONDAY, 9 JUNE – MORNING, 9:30 to 12:00

Attempt **FIVE** questions from **Section A** and **ONE** question from **Section B**.
Each question carries 50 marks.

WARNING: **Marks will be lost if all necessary work is not clearly shown.**

Answers should include the appropriate units of measurement, where relevant.

SECTION A
Answer FIVE questions from this section.

1. (a) A circle with centre $(-3, 2)$ passes through the point $(1, 3)$.
Find the equation of the circle.

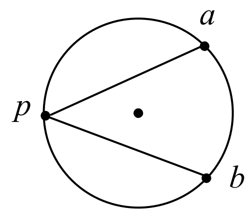
(b) **(i)** Prove that the equation of the tangent to the circle $x^2 + y^2 = r^2$ at the point (x_1, y_1) is $xx_1 + yy_1 = r^2$.

(ii) A tangent is drawn to the circle $x^2 + y^2 = 13$ at the point $(2, 3)$. This tangent crosses the x-axis at $(k, 0)$. Find the value of k.

(c) A circle passes through the points $a(8, 5)$ and $b(9, -2)$.
The centre of the circle lies on the line $2x - 3y - 7 = 0$.

(i) Find the equation of the circle.

(ii) p is a point on the major arc ab of the circle.
Show that $|\angle apb| = 45°$.



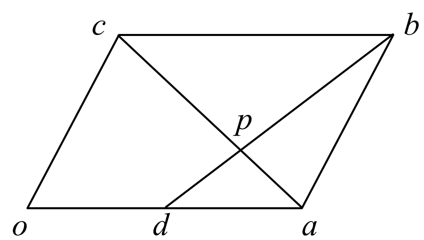
2. (a) Given that $\left|10\vec{i} + k\vec{j}\right| = \left|11\vec{i} - 2\vec{j}\right|$, find the two possible values of $k \in \mathbf{R}$.

(b) $\vec{x} = -\vec{i} + 3\vec{j}$, $\vec{y} = 4\vec{i} - 2\vec{j}$ and $\vec{z} = \vec{x} - t\vec{y}$, where $t \in \mathbf{R}$.

(i) Given that $\vec{x} \perp \vec{z}$, calculate the value of t.

(ii) Find the measure of $\angle xoy$, where o is the origin.

(c) $oabc$ is a parallelogram, where o is the origin.
d is the midpoint of $[oa]$ and $[db]$ cuts the diagonal $[ac]$ at p.



(i) Given that $\overrightarrow{ap} = k\,\overrightarrow{ac}$, where $k \in \mathbf{R}$,
express $\vec{p}$ in terms of $\vec{a}$, $\vec{c}$ and k.

(ii) Given that $\overrightarrow{bp} = l\,\overrightarrow{bd}$, where $l \in \mathbf{R}$, express $\vec{p}$ in terms of $\vec{a}$, $\vec{c}$ and l.

(iii) Hence find the value of k and the value of l.

3. **(a)** The parametric equations $x = 7t - 4$ and $y = 3 - 3t$ represent a line, where $t \in \mathbf{R}$. Find the Cartesian equation of the line.

(b) $a(2, 1)$, $b(10, 7)$, $c(14, 10)$ and $d(7, 1)$ are four points.

(i) Plot a, b, c and d on the co-ordinate plane.

(ii) Verify that $|ab| = 2|bc|$ and $|ab| = 2|ad|$.

(iii) Find a', b', c' and d', the respective images of a, b, c and d under the transformation $f: (x, y) \rightarrow (x', y')$, where $x' = x + y$ and $y' = x - 2y$.

(iv) Verify that $|a'b'| = 2|b'c'|$ but that $|a'b'| \neq 2|a'd'|$.

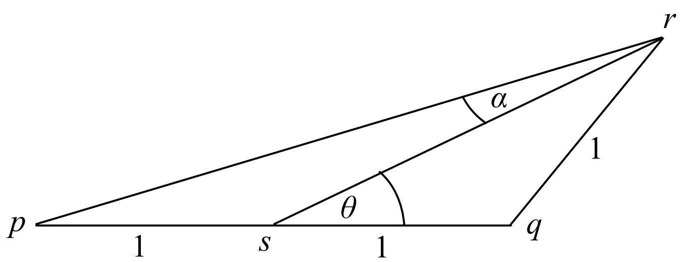
(c) Prove that the perpendicular distance from the point (x_1, y_1) to the line $ax + by + c = 0$ is $\dfrac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}}$.

4. **(a)** A and B are acute angles such that $\tan A = \dfrac{5}{12}$ and $\tan B = \dfrac{3}{4}$.
Find $\cos(A - B)$ as a fraction.

(b) **(i)** Show that $\dfrac{\sin 2A}{1 + \cos 2A} = \tan A$.

(ii) Hence, or otherwise, prove that $\tan 22\tfrac{1}{2}^\circ = \sqrt{2} - 1$.

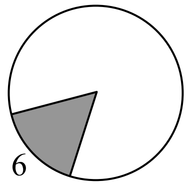
(c) In the triangle pqr, $|\angle rsq| = \theta^\circ$, $|\angle prs| = \alpha^\circ$, $|rq| = 1$, $|ps| = 1$ and $|sq| = 1$.



(i) Find $|sr|$ in terms of θ.

(ii) Hence, or otherwise, show that $\tan\theta = 3\tan\alpha$.

5. **(a)** In the shaded sector in the diagram, the arc is 6 cm long, and the angle of the sector is 0·75 radians.
Find the area of the sector.



(b) **(i)** Express $\sin 4x - \sin 2x$ as a product.

(ii) Find all the solutions of the equation

$$\sin 4x - \sin 2x = 0$$

in the domain $0^\circ \leq x \leq 180^\circ$.

(c) A triangle has sides of lengths a, b and c. The angle opposite the side of length a is A.

(i) Prove that $a^2 = b^2 + c^2 - 2bc\cos A$.

(ii) If a, b and c are consecutive whole numbers, show that

$$\cos A = \frac{a+5}{2a+4}.$$

6. **(a)** In a certain subject, the examination consists of a project, a practical test, and a written paper. The overall result is the weighted mean of the percentages achieved in these three components, using the weights 2, 3 and 5, respectively.
Michael scores 65% in the project and 80% in the practical.
What percentage mark must he get in the written paper in order to get an overall result of 70%?

(b) Solve the difference equation $u_{n+2} - 4u_{n+1} + u_n = 0$, where $n \geq 0$, given that $u_0 = 1$ and $u_1 = 2$.

(c) A bag contains discs of three different colours.
There are 5 red discs, 1 white disc and x black discs.
Three discs are picked together at random.

(i) Write down an expression in x for the probability that the three discs are all different in colour.

(ii) If the probability that the three discs are all different in colour is equal to the probability that they are all black, find x.

7. **(a)** Katie must choose five subjects from nine available subjects.
The nine subjects include French and German.

(i) How many different combinations of five subjects are possible?

(ii) How many different combinations are possible if Katie wishes to study German but not French?

(b) Four cards are drawn together from a pack of 52 playing cards.
Find the probability that

(i) the four cards drawn are the four aces

(ii) two of the cards are clubs and the other two are diamonds

(iii) there are three clubs and two aces among the four cards.

(c) The arithmetic mean of the three numbers x_1, x_2, x_3 is $\bar{x}$.
Let $d_1 = x_1 - \bar{x}$, $d_2 = x_2 - \bar{x}$ and $d_3 = x_3 - \bar{x}$.

(i) Show that $\sum_{r=1}^{3} d_r = 0$.

(ii) The standard deviation of the three numbers x_1, x_2, x_3 is σ.

Given any real number b, let $k^2 = \sum_{r=1}^{3} \frac{(d_r - b)^2}{3}$.

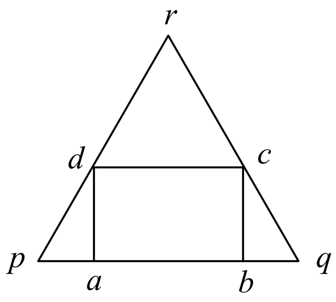
Show that $\sigma^2 = k^2 - b^2$.

SECTION B

Answer ONE question from this section.

8. **(a)** Use the ratio test to show that $\sum_{n=1}^{\infty} \frac{2^{3n+1}}{n!}$ is convergent.

(b) *pqr* is an equilateral triangle of side 6 cm.
abcd is a rectangle inscribed in the triangle as shown.
$|ab| = x$ cm and $|bc| = y$ cm.



(i) Express y in terms of x.

(ii) Find the maximum possible area of *abcd*.

(c) **(i)** Derive the Maclaurin series for $f(x) = \cos x$, up to and including the term containing x^4.

(ii) Hence, or otherwise, show that the first three non-zero terms of the Maclaurin series for $f(x) = \cos^2 x$ are $1 - x^2 + \frac{x^4}{3}$.

(iii) Use these to find an approximation for $\cos^2(0{\cdot}2)$, giving your answer correct to four decimal places.

9. **(a)** 20% of the items produced by a machine are defective. Four items are chosen at random. Find the probability that none of the chosen items is defective.

(b) Anne and Brendan play a game in which they take turns throwing a die. The first person to throw a six wins. Anne has the first throw.

(i) Find the probability that Anne wins on her second throw.

(ii) Find the probability that Anne wins on her first, second or third throw.

(iii) By finding the sum to infinity of a geometric series, or otherwise, find the probability that Anne wins the game.

(c) In order to test the hypothesis that a particular coin is unbiased, the coin is tossed 400 times. The number of heads observed is x. Between what limits should x lie in order that the hypothesis not be rejected at the 5% significance level?

10. **(a)** Let $x \oplus y = x + y - 4$, where $x, y \in \mathbf{Z}$.

(i) Find the identity element.

(ii) Find the inverse of x.

(iii) Determine whether $\oplus$ is associative on $\mathbf{Z}$.

(b) $(A, \circ)$ and $(B, *)$ are two groups. $A = \{k, l, m, n\}$ and $B = \{p, q, r, s\}$, and the Cayley tables for $(A, \circ)$ and $(B, *)$ are shown.

A:

$\circ$	k	l	m	n
k	l	k	n	m
l	k	l	m	n
m	n	m	k	l
n	m	n	l	k

B:

$*$	p	q	r	s
p	r	s	p	q
q	s	p	q	r
r	p	q	r	s
s	q	r	s	p

(i) Write down the identity element of $(A, \circ)$ and hence find a generator of $(A, \circ)$.

(ii) Find the order of each element in $(B, *)$.

(iii) Give an isomorphism ϕ from $(A, \circ)$ to $(B, *)$, justifying fully that it is an isomorphism.

11. **(a)** Find the coordinates of the point that is invariant under the transformation

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ 4 & -5 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 5 \\ 2 \end{pmatrix}.$$

(b) Prove that a similarity transformation maps the circumcentre of a triangle to the circumcentre of the image of the triangle.

(c) **(i)** E is the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ and f is the transformation

$$(x, y) \to (x', y'), \text{ where } x' = \frac{x}{a} \text{ and } y' = \frac{y}{b}.$$

Show that f maps E to the unit circle.

(ii) Hence, or otherwise, prove that the tangents drawn to an ellipse at the endpoints of a diameter are parallel to each other.

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