Introduction

Differential calculus has many real-life applications. It can be used to solve problems in engineering, biology, economics, astronomy and much more. Many functions are relationships between quantities (distance, money, volume etc) and time. Below are some examples

- **Mechanics.** Given a function describing displacement (distance) the derivative of the function at the point is velocity, the derivative of the velocity is acceleration.

- **Civil engineering, topography**. Given a function describing the height of a road, or the altitude of a mountain the derivative would be the gradient or slope of the road at that point.

- **Population growth**. Given a function describing the population of any biological organism the derivative with respect to time is the population growth rate. The growth rates of organisms be they human or other animals or bacterial is important to medicine, ecology and pharmaceuticals.

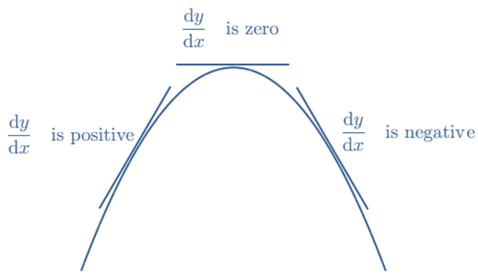
- **Economics.** The economic growth rate is the derivative of the GDP (gross domestic profit).

From previous chapters you should be able to differentiate linear, quadratic, polynomials, exponential, trigonometric, logarithmic, rational and inverse functions. We will now look at what this means when we are talking about real life examples.

Symbols and terminology		
The Function	f(x)	y value
The first derivative	f'(x)	$\frac{dy}{dx}$
The second derivative	f''(x)	$\frac{d2y}{dx2}$

Maximum and Minimum

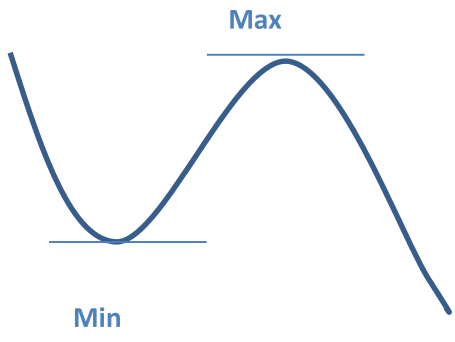
When we differentiate an equation the resulting equation or $\frac{dy}{dx}$ or f'(x) is the tangent to the curve at this point. Therefore, this first derivative is the rate of change of your original function.



On the right you can see the tangents to curve (first derivative) at different points on a function. Where f'(x) is greater than 0 the rate of change is positive, or the function is increasing. Once it reaches it maximum point where the tangent is parallel to the x axis the value of the first derivative is equal to zero. Then where f'(x) is less than 0 the function is decreasing.

This is also true for other functions such as polynomials and trigonometric equations

When dealing with a polynomial there may be a maximum and a minimum or several of each. In cubic equations there will be two local max/mins. In polynomials like x^4 we expect 3 max/mins.



In general

- If f' (x) > 0 for all x in the interval, then the function f is increasing.
- If f' (x) < 0 for all x in the interval, then the function f is decreasing.
- If f' (x) = 0 for all x in the interval, then the function f is constant. This will be the local maximum or minimum point.

Example 1

Find the local maximum/minimum point of following quadratic

$f(x) = x^2 - 2x - 3$

$\frac{dy}{dx} = 2x - 2$ *1) first we get the first derivative of the equation*

$\frac{dy}{dx} = 0$ *2) then we put the equation equal to 0 to identify where the rate of change is zero*

$2x - 2 = 0$

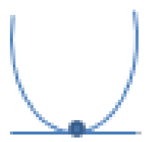
$2x = 2$

$x = 1$ *3) this is the x value, now we need to the y value to give us*

$f(1) = (1)^2 - 2(1) - 3$

$y = -4$

As this is a positive quadratic this point (1, -4) is the local minimum



Exercise 1

1) **Find the local maximum or minimum of the following**
   - (i) $x^2 + 4x - 6$
   - (ii) $2x^2 - 3x - 7$
   - (iii) $-x^2 + 2x - 1$
   - (iv) $-4x^2 + x - 13$

Local Max/Min in Real life

Being able to calculate the maximum or minimum of a function in real life can be very useful. It allows the highest or lowest values possible for functions to be calculated and can be used for optimisation.

Example 2

Example: A ball is thrown in the air. Its height at any time t is given by

$$h = 3 + 14t - 5t^2$$

What is its maximum height?

$\frac{dh}{dt} = 14 - 10t$ *1) first we get the first derivative of the equation*

$\frac{dh}{dt} = 0$ *2) then we put the equation equal to 0 to identify when the rate of change is zero*

$14 - 10t = 0$

$14 = 10t$

$1.4 = t$ *3) this means that the max/min happens at 1.4 seconds*

$h = 3 + 14(1.4) - 5(1.4)^2$

$h = 12.8$ *4) this means that 12.8 metres is the max and occurs after 1.4 seconds*

Example 3

The function $p = x^3 - 18x^2 + 105x - 88$ models the way the profit per item made, p pence, depends on x, the number produced in thousands.

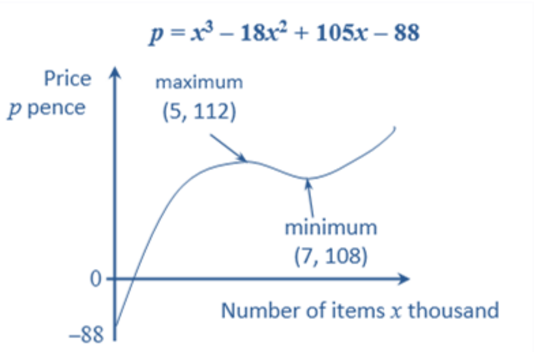
$\frac{dp}{dx} = 3x^2 - 36x + 105$ *1) first we get the first derivative of the equation*

$3x^2 - 36x + 105 = 0$ *2) then we put the equation equal to 0*

$x^2 - 12x + 35 = 0$

$(x - 5)(x - 7) = 0$

$x = 5$ or $x = 7$ *3) this is the x values, now*



When x is 5

$(5)^3 - 18(5)^2 + 105(5) - 88 = 112$

The point is (5, 112)

When x is 7

$(7)^3 - 18(7)^2 + 105(7) - 88 = 108$

The point is (7, 108)

When 5000 units are produced, the profit per item then being £1.12. The profit per item falls to £1.08 when 7000 are produced before rising again

To work out which of the of the points is the maximum and which is a minimum we use the second derivative. The rule is

- $f''(x) > 0$ = local minimum
- $f''(x) < 0$ = local maximum

In example 3 we can deduce that (5, 112) is the maximum and (7, 108) is the minimum based o they value but we can also test this using the above information such that

$p(x) = x^3 – 18x^2 + 105x – 88$

$p'(x) = 3x^2 – 36x + 105$

$p''(x) = 6x - 36$

When x = 5
6(5) – 36 = -6
Negative means (5, 112) is the max

When x = 7
6(7) – 36 = 6
Positive means (7, 108) is the min

Exercise 2

1) A plane initially flying at a height of 240 m dives to deliver some supplies. Its height after *t* seconds is

$$h = 8t^2 – 80t + 240 \text{ (m)}.$$

Find the plane's minimum height during the manoeuvre.

2) The velocity of a car, v m s^{-1} as it travels over a level crossing is modelled by $v = t^2 – 4t + 12$ for $0 \leq t \leq 4$ where *t* is the time in seconds after it reaches the crossing.

Find the car's minimum speed.

3) The closing price of a company's shares in pence is

$$p = 2x^3 – 12x^2 + 18x + 45 \quad \text{for } 0 \leq x \leq 5$$

where x is the number of days after the shares are released.

Find the maximum and minimum values of p.
Sketch a graph of p against x.

Functions can also be assigned to area and volume. This is particularly useful in manufacturing as you can optimise the volumes of receptacles like cans or tetra packs. It can also be applied to large 2-d and 3-d objects like bioreactors.

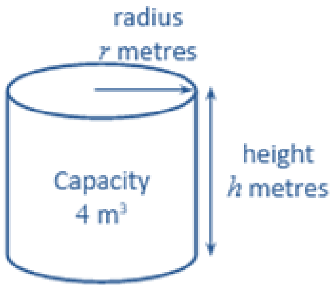
For volume or area questions most formulas can be found in the log tables supplied by the State Exam Commission.

Example 4

A cylindrical hot water tank is to have a capacity of 4 m^3.

Find the radius and height that would have the least surface area. *(least meaning minimum)*

The formulae for a cylinder are shown on the right.



Formulae for a cylinder:

Surface area $S = 2\pi r^2 + 2\pi rh$

Volume $V = \pi r^2 h = 4$

Substituting $h = \frac{4}{\pi r^2}$
into S gives a formula for the surface area
in terms of just one variable, r:

$S = 2\pi r^2 + 2\pi r\left(\frac{4}{\pi r^2}\right)$ *Substitute h into the formula*

$S = 2\pi r^2 + 8r^{-1}$ *Simplify*

$\frac{dS}{dr} = 4\pi r - 8r^{-2}$ *Differentiate*

$0 = 4\pi r - 8r^{-2}$ *Put equal to 0*

$4\pi r = \frac{8}{r^2}$ *Solve for r*

$r^3 = \frac{8}{4\pi} = 0.6366\ldots$

$r = \sqrt[3]{0.6366\ldots} = 0.860\ldots$

$\frac{d^2S}{dr^2} = 4\pi + 16r^{-3} = 4\pi + \frac{16}{r^3}$ is positive *Check if it's the min or max*

This implies minimum surface area.

Also

$h = \frac{4}{\pi r^2} = \frac{4}{\pi \times 0.860\ldots^2} = 1.72\ldots$ *Sub into the original formula*

The tank with minimum area has **radius 0.86 m** and **height 1.72 m**

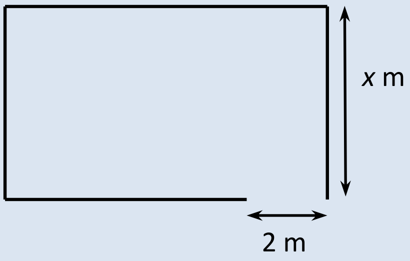
The minimum surface area can also be found:

$S = 2\pi r^2 + 2\pi rh$
$= 2\pi(0.86)^2 + 2\pi(0.86)(1.72)$

The minimum surface area is **13.9 m^2**

Exercise 3

1) A farmer has 100 metres of fencing to make a rectangular enclosure for sheep as shown. He will leave an opening of 2 metres for a gate.

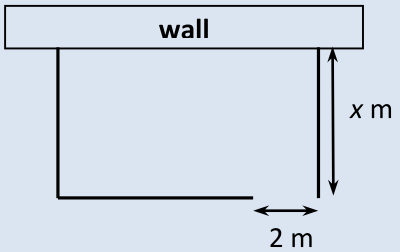


a **Show that the area of the enclosure is given by: $A = 51x - x^2$**

b **Find the value of x that will give the maximum possible area.**

c **Calculate the maximum possible area.**

2) A farmer has 100 metres of fencing to make a rectangular enclosure for sheep as shown. He will use an existing wall for one side of the enclosure and leave an opening of 2 metres for a gate.



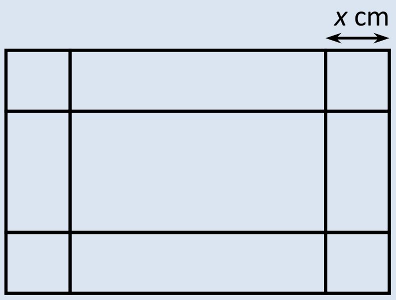
a **Show that the area of the enclosure is given by: $A = 102x - 2x^2$**

b **Find the value of x that will give the maximum possible area.**

c **Calculate the maximum possible area.**

3) An open-topped box is to be made by removing squares from each corner of a rectangular piece of card and then folding up the sides.

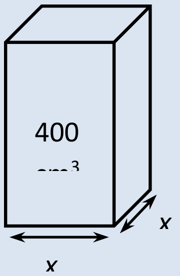
a **Show that, if the original rectangle of card measured 80 cm by 50 cm, and the squares removed from the corners have sides x cm long, then the volume of the box is given by: $V = 4x^3 - 260x^2 + 4000x$**



b **Find the value of x that will give the maximum possible volume.**

c **Calculate the maximum possible volume.**

A closed **tank is to have a square base and capacity 400 cm^3.**



a **Show that the total surface area of the container is given by:**

$$S = 2x^2 + \frac{1600}{x}$$

b **Find the value of x that will give the minimum surface area.**

c **Calculate the minimum surface area.**

Point of inflection

There are 3 types of stationary points:

- maximum points
- minimum points
- points of inflexion.

The point of inflection is where f''(x) = 0

An Inflection Point is where a curve changes from Concave upward to Concave downward (or vice versa)

Example 5

Find the point of inflection of $f(x) = 2 + 3x^2 - x^3$

$f'(x) = 6x - 3x^2$ *1) first we get the first derivative of the equation*

$f''(x) = 6 - 6x$ *2) then get the second derivative*

$0 = 6 - 6x$ *3) then put it equal to 0*

$6 = 6x$

$x = 1$

if x = 1 then

$2 + 3(1)^2 - (1)^3 = 4$

The point of inflection is (1, 4)

Exercise 4

1) **Find the point of inflection of the following**
   a) $y = x^3 + 3x^2 - 9x + 6$
   b) $y = 2x^3 - 3x^2 - 12x + 4$
   c) $y = x^3 - 3x - 5$

Calculus in mechanics and movement

As stated at the start of this section if you are given a function describing displacement (distance) the derivative of the function at the point is velocity, the derivative of the velocity is acceleration.

Simply put

Displacement → differentiate → **Velocity** → differentiate → **Acceleration**

These equations are differentiated with respect to time. It can be helpful to think about the units of each

Displacement (s)	**Height, distance, etc**	**S = $x^3 - 2x^2 + 5x - 2$**
Velocity (v)	**Rate of change of displacement with time ($\frac{m}{s}$)**	**V = $\frac{ds}{dt} = 3x^2 - 4x + 5$**
Acceleration (a)	**Rate of change of the rate of change($\frac{m}{s^2}$)**	**A = $\frac{dv}{dt} = 6x - 4$**

Note: When merging your knowledge of integration with differentiate it's worth noting that if you integrate acceleration you will get velocity and it you integrate velocity you will get displacement.

Example 6

A car is moving from one to another. It's distance from Town A is given by the equation

s = $t^3 - 4t^2 + 12t - 3$, where t is time in seconds and s is metres.

a) Find the velocity of the car after 5 seconds.

s = $t^3 - 4t^2 + 12t - 3$

$\frac{ds}{dt} = 3t^2 - 8t + 12$ = velocity or V *Differentiate once to get the rate of change in displacement (v)*

$\frac{ds}{dt} = 3(5)^2 - 8(5) + 12$

V(5) = 47m/s

b) Find the acceleration of the car after 10 minutes

$V = 3t^2 - 8t + 12$

$\frac{dV}{dt} = 6t - 8$ = acceleration or A *Differentiate again to get the rate of change in velocity (a)*

A(10) = 6(10) -8

= 52m/s^2

Exercise 5

1) **s(t) is a particle's position in a line after t seconds. Find the equations for the velocity and acceleration of the following functions. Then find the maximum velocity, where $0 \leq t \leq 10$.**
   a) **$s(t) = -3t^2 + 30t + 4$**
   b) **$s(t) = t^3 - 12t^2 + 45t – 2$**
   c) **$s(t) = 2t^3 - 27t^2 + 120t – 3$**
   d) **s(t) = t3 - 6t2 + 12t + 3**

2) **A particle moves in a straight line so that its position x(t) metres at time t seconds, relative to a fixed position O, is given by**

   $x(t) = t(t–4)^2$

   Find
   a) **the velocity at time t**
   b) **the values of t when the particle is at rest**
   c) **the acceleration after four seconds.**

Calculus in growth and decay

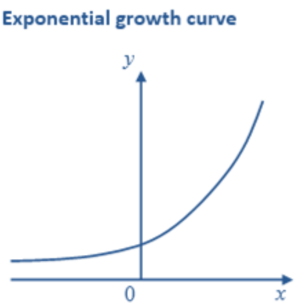
Almost all biological, chemical and physical processes undergo growth or decay. Understanding these processes and being able to describe and predict them is important for our understanding of the world, be that in predicating radioactive decay or the growth in populations. The most common ways things grow, or decay is exponentially. Differential calculus can be very useful in developing models of these different phenomenon and in allowing us to harness their power or curtail them.

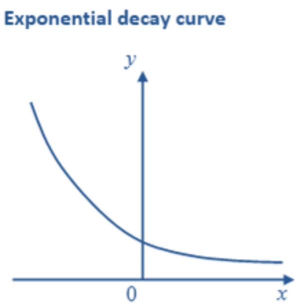
The number of a given subject, *N* at any time *t*, can be written as an exponential function of the form:

$N = Ae^{rt}$ where *A* and *r* are constants

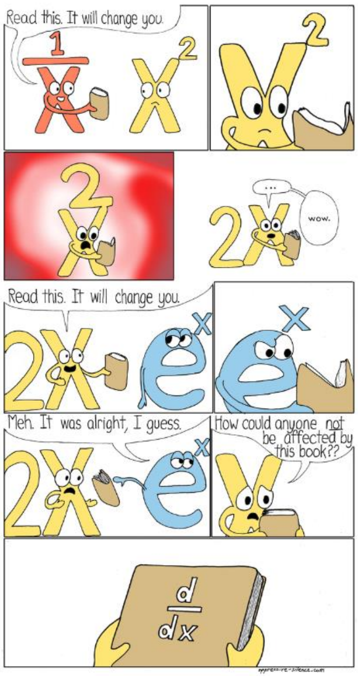
If r is positive is it a growth curve

If r is negative it is a decay curve





Differentiating e^x gives e^x (from log tables) and when the power has a coefficient that chain rule applies as from previous chapter on exponential functions.



Example 7

The Lady Bug (a woman's clothing chain) found that t days after the end of a sales promotion, the volume of sales was given by the function

$S(t) = 20,000(1+e^{-0.5t})$ in thousands of euros.

Find the rate of change in sales volume when t = 4.

$\frac{ds}{dt} = -40,000e^{-0.5t}$

$-40,000e^{-0.5(4)}$ = -5413.41 euros

This means sales volume is decreasing by 5413.41 per day at day 4.

Exercise 6

1) **The percentage of alcohol in a person's bloodstream t hours after drinking 8 fluid oz of whiskey is given by**

$$A(t) = 0.23te^{0.4t} \qquad (0 \leq t \leq 12)$$

   a) **What is the percentage of alcohol in a person's bloodstream after half an hour?**
   b) **After 8 hr?**
   c) **How fast is the percentage of alcohol in a person's bloodstream changing after 8 hours?**

2) **The monthly demand for a certain brand of perfume is given by the demand equation**

$$p = 100e^{-0.0002x} + 150$$

where p denotes the retail unit price (in dollars) and x denotes the quantity (in 1-oz bottles) demanded.

   a) **Find the rate of change of the price per bottle when x = 1000 and when x = 2000.**
   b) **What is the price per bottle when x 1000?**

3) **It has been estimated that the total production of oil from a certain oil well is given by**

$$T(t) = 1000(t + 10)e\text{-}^{0.1t} + 10,000$$

thousand barrels t years after production has begun.

   a) **Determine the year when the oil well will be producing at maximum capacity.**

Example 8

Under ideal laboratory conditions, the number of bacteria in a culture grows in accordance with the law

$Q(t) = Q_0e^{kt,}$

where Q_0 denotes the number of bacteria initially present in the culture, k is some constant determined by the strain of bacteria under consideration, and t is the elapsed time measured in hours.

Suppose 10,000 bacteria are present initially in the culture and 60,000 present 2 hours later.

a. How many bacteria will there be in the culture at the end of 4 hours?

b. What is the rate of growth of the population after 4 hours?

a) We are given that $Q(0) = Q_0 = 10{,}000$,

$Q(t) = 10{,}000e^{kt}$.

$60{,}000 = 10{,}000e^{2k}$ Next, the fact that 60,000 bacteria are present 2 hours later

$e^{2k} = 6$ Taking the natural logarithm on both sides of the equation

$\ln e^{2k} = \ln 6$

$2k = \ln 6$

$k = 0.8959$ (approx.)

$Q(4) = 10{,}000e^{0.8959(4)}$ = 360,029 bacteria at the end of 4 hours

b) To get the rate we differentiate the function

$Q'(t) = 8959e^{0.8959(t)}$

$8959e^{0.8959(4)}$ = 322559 or or approximately 322,550 bacteria per hour

Exercise 7

1) The Metro Department Store found that t week after the end of a sales promotion the volume of sales was given by a function of the form

$$S(t) = B + Ae^{-kt} \qquad (0 \le t \le 4)$$

where B = 50,000 and is equal to the average weekly volume of sales before the promotion. The sales volumes at the end of the first and third weeks were $83,515 and $65,055, respectively. Assume that the sales volume is decreasing exponentially.

a) Find the decay constant k.
b) Find the sales volume at the end of the fourth week.
c) How fast is the sales volume dropping at the end of the fourth week?

2) The Universal Instruments Company found that the monthly demand for its new line of Galaxy Home Computers t months after placing the line on the market was given by

$$D(t) = 2000 - 1500e^{-0.05t} \qquad (t \ge 0)$$

Graph this function over the first 3 years and answer the following questions.

a) What is the demand after 1 month and after 3 years?
b) At what level is the demand expected to stabilize?
c) Find the rate of growth of the demand after the tenth month.

Differentiation of Trigonometric Functions

Trigonometric functions such as sine, cosine and tangent can be used in many facets of life. The sine and cosine functions are used to model periodic phenomena in nature, such as waves, tides and signals, essentially any phenomenon controlled by circle or oscillatory motion. This also allows these functions to be used in economics, ecology, electrical engineering and anatomy. From previous chapters you should be aware of the rules surrounding differentiating Trigonometric Functions. To refresh

$\frac{d(sinx)}{dx} = \cos x$ **and** $\frac{d(cosx)}{dx} = -sinx$

But also remember if there is a coefficient or power on the x this must also be differentiated and brought forward.

$2Cos4x^2 = (8x)(2)(-sin4x^2)$

$= (-16x)sin4x^2$

Other than the special cases of differentiation all other laws of calculus still apply. If we differentiate, we get the rate of change and we can work out local max/min etc in the normal way.

The general form for a sine function is

y = a sin (bx) + c

a is the ***amplitude*** of the sine curve - the amplitude of a sine curve is its height.

b is the ***period*** of the sine curve - The period of the sine curve is the length of one cycle of the curve. This can be calculated by dividing the 2π by the length of one cycle.

c is the ***phase*** shift of the sine curve - The phase shift of a sine curve is how much the curve shifts from zero.

Cosine is the same y = a cos (bx)+ c

Sine starts at 0 and Cosine starts at 1. This is important in informing which function you should use.

In these calculations remember your calculator must be set to radians.

Remember Trigonometric Functions can be on either paper 1 or 2!!

Example 9

For example, on a winter day, the high tide in Boston, Massachusetts, occurred at midnight. To determine the height of the water in the harbour, use the equation

H(t) = $\mathbf{4.8sin\frac{\pi}{6}(t+3)+5.1}$ $\mathbf{0 \leq t \leq 2\pi}$

where t represents the number of hours since midnight.

What is the heights of the tide at its maximum and minimum and when does it occur?

H(t) = $4.8sin\frac{\pi}{6}(t+3)+5.1$

H' (t) = $(\frac{\pi}{6})4.8cos\frac{\pi}{6}(t+3)$ = 0 To find the maximum we differentiate and put equal to 0

$$cos\frac{\pi}{6}(t+3)=0$$

When is cosine = 0 in the domain? At $\frac{\pi}{2}$ and $\frac{3\pi}{2}$

$\frac{\pi}{6}(t+3)=\frac{\pi}{2}$

$2\pi(t+3)=6\pi$
2t + 6 = 6
T = 0

$\frac{\pi}{6}(t+3)=\frac{3\pi}{2}$

$2\pi(t+3)=18\pi$
2t + 6 = 18
2t = 18
T = 6

At t = 0 the height is

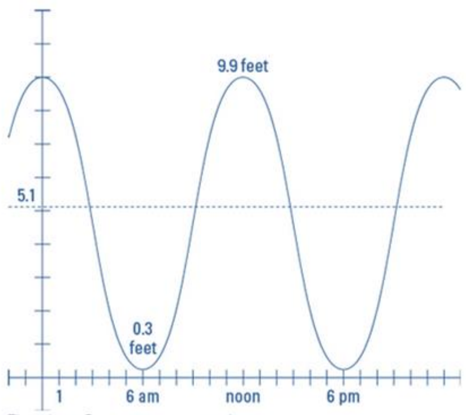
H(t) = $4.8sin\frac{\pi}{6}(0+3)+5.1$

= 9.9m

At t = 6 the height is

H(t) = $4.8sin\frac{\pi}{6}(6+3)+5.1$

= 0.3m



The tides in Boston on one wintry day.

At time = 0 we expect our first maximum of a height 9.9m but from our model we know this will repeat as the function is oscillating (see above diagram)

At time = 6 we expect our first minimum of a height 0.3m but from our model we know this will repeat as the function is oscillating (see above diagram)

Every 12 hours from our model as $\frac{2\pi}{\frac{\pi}{6}}$ = 12

Exercise 8

1) Suppose that the population (in thousands) of a certain kind of insect after tt months is given by the following formula

$P(t)=3t + \sin(4t) + 100$

Determine the minimum and maximum population in the first 4 months.

2) Depth of water is 6m at low tide and 16m at high tide, which is 6 hours later. Assuming the motion of the water is a simple harmonic, draw a graph to show how the water varies with time over a period of 12 hours, commencing at low tide. Use calculus to determine the height at high and low tide. Confirm your answer from the graph.

3) The voltage, V, in volts, in an electrical outlet is given as a function of time, t, in seconds, by the function
$V = 156 \cos (120 \pi t)$.
a) Give an expression for the rate of change of voltage with respect to time.
b) Is the rate of change ever zero? Explain.
c) What is the maximum value of the rate of change?

4) The amount of daylight a location on Earth receives on a given day of the year can be modelled by a sinusoidal function. The amount of daylight that Windsor, Ontario will experience in 2007 can be modelled by the function

$D(t) = 12.18 + 3.1 \sin(0.017t – 1.376)$

where t is the number of days since the start of the year.
a) On January 1^{st}, how many hours of daylight does Windsor receive?
b) The summer solstice is the day on which the maximum amount of daylight will occur. On what day of the year would this occur?
c) What is the maximum amount of daylight Windsor receives?

Related rates of change

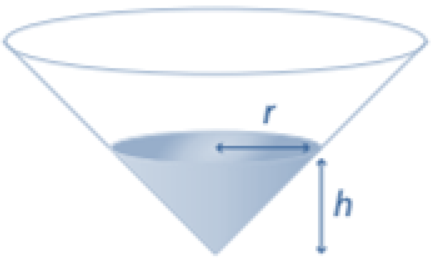
Related rates of change are simply an application of the chain rule. In related-rate problems, you find the rate at which some quantity is changing by relating it to other quantities for which the rate of change is known.

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dy}$$

The key to answering these questions is to identify the differential equation you are looking for and what you already have.

Example 9

An upturned cone with semi-vertical angle 45◦ is being filled with water at a constant rate of 30 cm^3 per second.



When the depth of the water is 60 cm, find the rate at which the depth of the water, is increasing

The volume V of the water in the cone is given by

$V = \frac{1}{3}\pi r^3$ where r is h in this question

In part a we need $\frac{dh}{dt}$ and we have $\frac{dV}{dt}$ from the start of the question and we can get $\frac{dV}{dh}$ and invert it from the about equation

$\frac{dh}{dt} = \frac{dh}{dV} \cdot \frac{dV}{dt}$

$= \frac{1}{\pi h^2} \cdot 30$

$= \frac{30}{\pi (60)^2}$

$$\frac{dh}{dt} = \frac{1}{120\pi}\ cm/s$$

Exercise 9

1) The formula for the area of a circle is $A = \pi r^2$ where r is the radius of the circle. Suppose a circle is expanding, meaning that both the area A and the radius r (in inches) are expanding.
   a) Suppose r=2 - $\frac{100}{(x+7)^2}$ where t is time in seconds. Use $\frac{dA}{dt} = \frac{dA}{dr} \cdot \frac{dr}{dt}$ find the rate at which the area is expanding.
   b) Use a. to find the rate at which the area is expanding at t=4 s

2) A spherical balloon is being inflated in such a way that the radius increases at a constant rate of 2 inches per second.
   a) At what rate is should air be pumped into the balloon when the radius is 1 foot?
   b) When the radius is 20 feet?

3) A spherical raindrop is formed by condensation. In an interval of 10 sec. its volume increases at a constant rate from 0.010mm^3 to 0.500mm^3.

   a) Find the rate at which the surface area of the raindrop is increasing, when its radius is 1.0mm